

Estimates of densities for Lévy processes with lower intensity of large jumps

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Abstract

We obtain general lower estimates of transition densities of jump Lévy processes. We use them for processes with Lévy measures having bounded support, processes with exponentially decaying Lévy measures for large times and for processes with high intensity of small jumps for small times.

1 Introduction

Let $d \in \{1, 2, \dots\}$ and ν be a symmetric Lévy measure on \mathbb{R}^d , i.e.,

$$(1) \quad \int_{\mathbb{R}^d} (1 \wedge |y|^2) \nu(dy) < \infty,$$

and $\nu(-D) = \nu(D)$ for every Borel set $D \subset \mathbb{R}^d$. We always assume also that $\nu(\mathbb{R}^d) = \infty$.

We consider the convolution semigroup of probability measures $\{P_t, t \geq 0\}$ with the Fourier transform $\mathcal{F}(P_t)(\xi) = \int_{\mathbb{R}^d} e^{i\xi \cdot y} P_t(dy) = \exp(-t\Phi(\xi))$, where

$$\Phi(\xi) = \int_{\mathbb{R}^d} (1 - \cos(\xi \cdot y)) \nu(dy), \quad \xi \in \mathbb{R}^d.$$

There exists a Lévy process $\{X_t, t \geq 0\}$ corresponding to $\{P_t, t \geq 0\}$, i.e., P_t is the transition function of X_t .

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We denote

$$\Psi(r) = \sup_{|\xi| \leq r} \Phi(\xi), \quad r > 0.$$

It follows directly from the definition that $\Psi(|\xi|) \geq \Phi(\xi)$ for $\xi \in \mathbb{R}^d$. An opposite inequality $\Psi(|\xi|) \leq c\Phi(\xi)$ holds also in many typical examples but is not true in general.

We will often use the following estimate obtained in Proposition 1 in [17] (see also Lemma 6 in [12])

$$(2) \quad L_0 H(r) \leq \Psi(r) \leq 2H(r), \quad r > 0,$$

where

$$H(r) = \int (1 \wedge r^2 |y|^2) \nu(dy),$$

and L_0 depends only on the dimension d .

We note that Ψ is continuous and nondecreasing and $\sup_{r>0} \Psi(r) = \infty$, since $\nu(\mathbb{R}^d) = \infty$ (it follows easily from (2)). Let $\Psi^{-1}(s) = \sup\{r > 0 : \Psi(r) = s\}$ for $s \in (0, \infty)$ so that $\Psi(\Psi^{-1}(s)) = s$ and $\Psi^{-1}(\Psi(s)) \geq s$ for $s > 0$. Define

$$h(t) = \frac{1}{\Psi^{-1}\left(\frac{1}{t}\right)}, \quad t > 0.$$

We often use the following condition which is satisfied under mild assumptions on Φ (see Lemma 5 in [17] or Lemma 5 in [18]).

(A1) *There exist constants $M_0 > 0$, and $t_p \in (0, \infty]$ such that*

$$\int_{\mathbb{R}^d} e^{-t\Phi(\xi)} |\xi| d\xi \leq M_0 (h(t))^{-d-1}, \quad t \in (0, t_p).$$

We note that (A1) yields in particular that $\nu(\mathbb{R}^d) = \infty$ and the existence of the transition densities p_t of P_t for all $t > 0$.

The main results of the present paper are the following two lower estimates of the transition densities. They contain universal minimal bounds for jump Lévy processes.

THEOREM 1.1. *For every symmetric Lévy measure ν such that (A1) holds with $t_p = \infty$ there exists positive constants $c_1 - c_4$, such that*

$$p_t(x) \geq c_1 h(t)^{-d} e^{\frac{-c_2 |x|^2}{t}}, \quad t > c_3, |x| \leq c_4 t,$$

where p_t is the density of P_t .

Let $\nu = \nu_s + \nu_c$ where ν_s and ν_c are singular and continuous part of ν with respect to the Lebesgue measure on $\mathbb{R}^d \setminus \{0\}$, respectively, and let $\frac{d\nu_c}{dm}$ denote the Radon-Nikodym derivative of ν_c .

THEOREM 1.2. Assume that **(A1)** holds and there exists $r_0 > 0$ such that

$$\inf_{0 < |y| < r_0} \frac{d\nu_c}{dm}(y) > 0.$$

Then there exist constants $c_1 - c_4$, such that

$$p_t(x) \geq c_1 e^{-c_2 |x| \log\left(\frac{c_3 |x|}{t}\right)},$$

for $t \in (0, t_p)$, and $|x| \geq \max\{r_0, c_4 t\}$.

We prove the theorems in Section 3. Note that explicit values of the constants and also more specific estimates can be found in Lemma 3.1, Lemma 3.2 and Lemma 3.3 in Section 3. We emphasise also that lower bounds obtained for transition densities in previous papers depend usually on the local behaviour of the Lévy measure ν (see [30],[31]) or hold only for isotropic processes ([2],[7]). In particular, Proposition 2.1 below gives the lower bound in terms of the Lévy measure: $p_t(x) \geq c_1 t h(t)^{-d} \nu(B(x, c_2 h(t)))$, for $|x| > c_3 h(t)$ and $t \in (0, t_p)$, and the both above theorems deliver useful estimates even in regions on which ν is not supported.

Although the above estimates of transition densities hold for wide class of Lévy processes one can hardly expect that they are optimal for processes with heavy tails of the Lévy measure since it is known that $\nu(dx) = g(x)dx$ is a vague limit of measures $P_t(dx)/t = (p_t(x)/t)dx$ as $t \rightarrow 0^+$ outside the origin, and in fact the both functions $p_t(x)$ and $tg(x)$ share typically the same asymptotic properties for such processes for small times (see results and discussions in [17], [18]). In particular for α -stable processes with $\nu(dx) \asymp |x|^{-d-\alpha}dx$ we have $p_t(x) \asymp t^{-d/\alpha}(1 + t^{-1/\alpha}|x|)^{-d-\alpha}$. Therefore the above estimates are useful mainly for processes with truncated jumps or with exponentially decaying intensity of jumps and large times where the asymptotic of $p(t)$ and $tg(x)$ can differ significantly. In the next sections we give some applications and show that the above results are optimal or close to optimal for the considered processes using existing upper estimates.

The first natural application are processes with truncated Lévy measures. In [8] the authors obtained both side estimates of transition densities for processes with truncated isotropic stable Lévy measure. Here we extend the results of [8] to much wider class of processes with Lévy measure with bounded support and not necessarily absolutely continuous. The lower estimates which follow easily from the above inequalities and Proposition 2.1 are presented in Section 4 in Theorem 4.1. Next we prove upper estimates in Lemma 4.2 and Theorem 4.3. Our method is based on the results of [21] where the authors complement in very useful way the known results of Carlen, Kusuoka, and Stroock ([5]). Assuming additionally that the Lévy measure is absolutely continuous we obtain more precise following estimates which can be regarded as the third main result of the present paper. We note that similar estimate was announced (without a proof) in Theorem 1.4 of [6].

We will use here the following condition on a function $f : (0, r_0] \rightarrow (0, \infty)$.

(A2) *There exist constants $M_1, M_2 \geq 0$ and $d < \beta_1 \leq \beta_2 < d + 2$, such that*

$$M_1 \left(\frac{R}{r}\right)^{\beta_1} \leq \frac{f(r)}{f(R)} \leq M_2 \left(\frac{R}{r}\right)^{\beta_2}, \quad r_0 \geq R \geq r > 0.$$

We will use the notation $f \asymp g$ to indicate that there exist constants c_1, c_2 such that $c_1 g \leq f \leq c_2 g$.

THEOREM 1.3. *Assume that $\text{supp}(\nu) \subset B(0, r_0)$, ν is symmetric and absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d \setminus \{0\}$ with a density $\bar{\nu}$ and there exists a nonincreasing function $f : (0, r_0] \rightarrow (0, \infty]$ such that*

$$\bar{\nu}(x) \asymp f(|x|), \quad 0 < |x| < r_0,$$

where f satisfies (A2) and $\kappa = \inf_{s \in (0, r_0]} f(s) > 0$.

Then P_t is for every $t > 0$ absolutely continuous with a density p_t which satisfies the following estimates.

1. *There exists η_* such that for $|x| \leq \eta_* h(t)$ we have*

$$p_t(x) \asymp h(t)^{-d}.$$

2. *There exists C^* such that for $\eta_* h(t) \leq |x| \leq r_0$, $t \leq t_1$ we have*

$$p_t(x) \asymp t f(|x|),$$

where $t_1 = r_0 / C^*$.

3. *There exist positive constants $c_1 - c_4$ such that*

$$c_1 h(t)^{-d} \exp \left\{ -\frac{c_2 |x|^2}{t} \right\} \leq p_t(x) \leq c_3 h(t)^{-d} \exp \left\{ -\frac{c_4 |x|^2}{t} \right\},$$

for $\eta_* h(t) \leq |x| \leq C^* t$, $t \geq t_1$.

4. *There exist positive constants $c_5 - c_{10}$ such that*

$$c_5 \exp \left\{ -c_6 |x| \log \left(\frac{c_7 |x|}{t} \right) \right\} \leq p_t(x) \leq c_8 \exp \left\{ -c_9 |x| \log \left(\frac{c_{10} |x|}{t} \right) \right\},$$

for $|x| \geq r_0 \vee C^* t$, $t > 0$.

The second interesting application are Lévy processes with exponentially decaying Lévy measure which we discuss in Section 5. We extend here the results obtained previously in [6] and [18]. The sharp estimates obtained in [18] hold only for small times whereas the results of [6] contain only absolutely continuous Lévy measures. Here we obtain in Theorem 1.4 both side estimates for large times and not necessarily absolutely continuous Lévy measures.

THEOREM 1.4. *Let*

$$\nu(A) \asymp \int_0^\infty \int_{\mathbb{S}} \mathbb{1}_A(s\theta) s^{-1-\alpha} (1+s)^\kappa e^{-ms^\beta} ds \mu(d\theta),$$

where μ is bounded, symmetric and nondegenerate measure on the unit sphere \mathbb{S} , $m > 0$, $\beta \in (0, 1]$, $\alpha \in (0, 2)$, $\kappa \in (-\infty, 1 + \alpha]$. Then there exist constants $c_1 - c_6, \eta, t_0$ such that

$$(3) \quad p_t(x) \leq c_1 t^{-d/2} \left(e^{\frac{-c_2|x|^2}{t}} + e^{\frac{-m|x|^\beta}{2.4\beta}} \right),$$

for $x \in \mathbb{R}^d$, $t > t_0$, and

$$(4) \quad p_t(x) \geq c_3 t^{-d/2} \left(e^{\frac{-c_4|x|^2}{t}} + t\nu(B(x, c_5\sqrt{t})) \right), \quad \eta\sqrt{t} \leq |x| \leq c_6 t, \quad t > t_0.$$

In particular, if

$$(5) \quad \nu(dx) \asymp |x|^{-d-\alpha} (1+|x|)^\kappa e^{-m|x|^\beta} dx, \quad x \in \mathbb{R}^d \setminus \{0\},$$

then there exist $c_7 - c_9$ such that

$$(6) \quad c_7 t^{-d/2} \left(e^{\frac{-c_8|x|^2}{t}} + e^{-c_9|x|^\beta} \right) \leq p_t(x) \leq c_1 t^{-d/2} \left(e^{\frac{-c_2|x|^2}{t}} + e^{\frac{-m|x|^\beta}{2.4\beta}} \right), \quad x \in \mathbb{R}^d, \quad t > t_0.$$

The last application is given in Section 6. We do not assume here anything (except of (1)) on the behavior of ν outside of the ball $B(0, 1)$ and we consider the processes with high intensity of small jumps, i.e., such that $\nu(dx) \asymp |x|^{-d-2} \left[\log \left(\frac{2}{|x|} \right) \right]^{-\beta} dx$, for $|x| < 1$, where $\beta > 1$. We extend the results obtained previously in [23] and [17]. In this case sharp estimates for large times were already known. We investigate here the difficult case of small t and using Lemma 3.2 we get a new lower bound. This estimate seems to be optimal in view of new results obtained in [24] for a particular case of subordinated Brownian motion.

Let us also mention other related results. Estimates of transition densities for stable Lévy processes has been studied, e.g., in [1, 26, 13, 14, 10, 11, 9, 31, 4]. Recent papers [29, 30, 20, 21, 16, 19] contain the estimates for more general classes of Lévy processes, including tempered processes with intensities of jumps lighter than polynomial. The paper [2] deals with estimates of densities for isotropic unimodal Lévy processes, while the papers [23, 16] discusses the processes with higher intensity of small jumps, remarkably different than stable one. In [6, 7, 16] the authors investigate the case of more general, non-necessarily space homogeneous, symmetric jump Markov processes with jump intensities dominated by those of isotropic stable processes. Estimates of kernels for processes which are solutions of SDE driven by Lévy processes were obtained in [25]. For estimates of derivatives of Lévy densities we refer the reader to [28, 3, 27, 16, 22, 19]. In [15] an interesting geometric interpretation of the transition densities for symmetric Lévy processes was given.

2 Preliminaries

For a set $A \subset \mathbb{R}^d$ we denote $\delta(A) = \text{dist}(0, A) = \inf\{|y| : y \in A\}$ and $\text{diam}(A) = \sup\{|y - x| : x, y \in A\}$. By $\mathcal{B}(\mathbb{R}^d)$ we denote Borel sets in \mathbb{R}^d .

General estimates of the densities at the origin were obtained in [17]. It follows from Lemma 6 and 7 in [17] that if **(A1)** holds then there exist constants $c_1 = c_1(d)$, $c_2 = c_2(d, M_0)$, $\theta = \theta(d, M_0)$ such that

$$(7) \quad c_1 (h(t))^{-d} \leq p_t(x) \leq c_2 (h(t))^{-d} \quad \text{for } |x| < \theta h(t), t \in (0, t_p).$$

Lower estimates of densities by the Lévy measure were also obtained in [17]. We include here a modified version of Theorem 2 of [17]. The proof differs only in a few details and we give it in the Appendix.

PROPOSITION 2.1. *If **(A1)** holds then for every $\eta > 0$ there exist constants $L_1 = L_1(d, \eta, M_0)$, $L_2 = L_2(d, \eta, M_0) < \eta$ such that*

$$(8) \quad p_t(x) \geq L_1 t (h(t))^{-d} \nu(B(x, L_2 h(t))) \quad \text{for } |x| \geq \eta h(t), t \in (0, t_p).$$

The following proposition was proved in [17], Theorem 1.

PROPOSITION 2.2. *Assume that ν is a symmetric Lévy measure such that*

$$(9) \quad \nu(A) \leq M_3 f(\delta(A)) [\text{diam}(A)]^\gamma, \quad A \in \mathcal{B}(\mathbb{R}^d),$$

where $\gamma \in [0, d]$, and $f : [0, \infty) \rightarrow [0, \infty]$ is nonincreasing function satisfying

$$(10) \quad \int_{|y| > r} f\left(s \vee |y| - \frac{|y|}{2}\right) \nu(dy) \leq M_4 f(s) \Psi\left(\frac{1}{r}\right), \quad s > 0, r > 0,$$

for some constants $M_3, M_4 > 0$. If **(A1)** holds then there exist constants $c_1 = c_1(d, M_0, M_3, M_4)$, $c_2 = c_2(d, M_0)$, $c_3 = c_3(d, M_0)$ such that

$$p_t(x) \leq c_1 (h(t))^{-d} \min \left\{ 1, t [h(t)]^\gamma f(|x|/4) + e^{-c_2 \frac{|x|}{h(t)} \log\left(1 + \frac{c_3 |x|}{h(t)}\right)} \right\},$$

$$x \in \mathbb{R}^d, t \in (0, t_p).$$

3 Lower estimates

The following lemma contains a lower estimate of densities p_t in terms of a function F which is a lower bound for $\Psi(s)/s$.

LEMMA 3.1. *Assume that **(A1)** holds and there exists a strictly increasing continuous function $F : [0, \infty) \rightarrow [0, \infty)$ such that $F(0) = 0$, $\lim_{s \rightarrow \infty} F(s) = \infty$ and*

$$(11) \quad F(s) \leq \frac{\Psi(s)}{s}, \quad \text{for } s \in [0, \infty).$$

Then there are constants $c_i = c_i(d, M_0)$, $i = 1, 2$, such that for every $\eta \in (0, \theta)$, where θ is the constant from (7), we have

$$p_t(x) \geq c_1 h(t)^{-d} e^{-c_2 |x| F^{-1}(2|x|/(\eta t))/\eta},$$

for $t \in (0, t_p)$ and $x \in \mathbb{R}^d$.

We may also weaken the assumptions obtaining estimates on smaller domain. We give only the proof of Lemma 3.2 since the proof of Lemma 3.1 differs only in few details (one can just put $s_0 = \infty$ here).

LEMMA 3.2. Assume that **(A1)** holds and there exists a constant $s_0 \in (0, \infty)$ and a strictly increasing continuous function $F : [0, s_0] \rightarrow [0, \infty)$ such that $F(0) = 0$, and

$$(12) \quad F(s) \leq \frac{\Psi(s)}{s}, \quad \text{for } s \leq s_0.$$

Then there are constants $c_i = c_i(d, M_0)$, $i = 1, 2$, such that for every $\eta \in (0, \theta)$, where θ is the constant from (7), we have

$$(13) \quad p_t(x) \geq c_1 h(t)^{-d} e^{-c_2 |x| F^{-1}(2|x|/(\eta t))/\eta},$$

for $t \in \left(\frac{1}{s_0 F(s_0/2)}, t_p\right)$ and $|x| < \frac{\eta t F(s_0/2)}{2}$.

Proof. Let $t \in \left(\frac{1}{s_0 F(s_0/2)}, t_p\right)$, $\eta \in (0, \theta)$ and $x \in \mathbb{R}^d$ be such that $0 < |x| < \eta t F(s_0/2)/2$.

We first assume that $\frac{4|x|}{\eta} F^{-1}\left(\frac{2|x|}{\eta t}\right) \geq 1$ and let $n \in \mathbb{N}_0$ be such that

$$2^n \leq \frac{4|x|}{\eta} F^{-1}\left(\frac{2|x|}{\eta t}\right) < 2^{n+1}.$$

We have $F^{-1}\left(\frac{2|x|}{\eta t}\right) < \frac{\eta 2^{n+1}}{4|x|}$, and by (12) we obtain

$$\frac{2|x|}{\eta t} < F\left(\frac{\eta 2^{n+1}}{4|x|}\right) = F\left(\frac{\eta 2^n}{2|x|}\right) \leq \Psi\left(\frac{\eta 2^n}{2|x|}\right) \frac{2|x|}{\eta 2^n},$$

since $\frac{\eta 2^n}{2|x|} \leq 2F^{-1}\left(\frac{2|x|}{\eta t}\right) < s_0$, hence $\frac{2^n}{t} < \Psi\left(\frac{\eta 2^n}{2|x|}\right)$ and $\Psi^{-1}\left(\frac{2^n}{t}\right) \leq \frac{\eta 2^n}{2|x|}$ which gives

$$\frac{|x|}{2^n} \leq \frac{1}{2} \eta h(t/2^n) \leq \frac{1}{2} \theta h(t/2^n).$$

Let $k = 2^n$. It follows from (7) that

$$(14) \quad p_{t/k}(y) \geq c_1 h(t/k)^{-d}, \quad \text{for } |y| < \theta h(t/k).$$

Having the above preparation we can use now the standard method which was used, e.g., in the proof of Theorem 3.6 in [8]. Let $0 = x_0, x_1, \dots, x_{k-1}, x_k = x$ be such that $x_i = (i/k)x$. We have $|x_{i+1} - x_i| = \frac{|x|}{k} \leq \frac{1}{2}\eta h(t/k) \leq \frac{1}{2}\theta h(t/k)$. Let $B_i = B(x_i, \frac{\theta}{4}h(t/k))$. Using the semigroup property of p_t and (14) we get

$$\begin{aligned}
p_t(x) &= \int \dots \int p_{t/k}(y_1)p_{t/k}(y_2 - y_1)\dots p_{t/k}(x - y_{k-1}) dy_1 dy_2 \dots dy_{k-1} \\
&\geq \int_{B_1} \dots \int_{B_{k-1}} p_{t/k}(y_1)p_{t/k}(y_2 - y_1)\dots p_{t/k}(x - y_{k-1}) dy_1 dy_2 \dots dy_{k-1} \\
&\geq \left(c_1 h(t/k)^{-d}\right)^k \left(\omega_d \left(\frac{\theta}{4}h(t/k)\right)^d\right)^{k-1} \\
&= h(t/k)^{-d} \left(c_1 \omega_d \left(\frac{\theta}{4}\right)^d\right)^k \left(\omega_d \left(\frac{\theta}{4}\right)^d\right)^{-1} \\
&= c_2 h(t/k)^{-d} e^{-c_3 k} \geq c_2 h(t)^{-d} e^{-4c_3 |x| F^{-1}(2|x|/(\eta t))/\eta},
\end{aligned}$$

where $c_1 = c_1(d)$ and $c_2 = c_2(d, M_0), c_3 = c_3(d, M_0)$. Let now $\frac{4|x|}{\eta} F^{-1}\left(\frac{2|x|}{\eta t}\right) < 1$. The function $g(s) = \frac{4s}{\eta} F^{-1}\left(\frac{2s}{\eta t}\right)$ is strictly increasing, continuous and $\sup_{s \in [0, \eta t F(s_0)/2]} g(s) = g(\eta t F(s_0)/2) = 2t s_0 F(s_0) > 1$ (since $t > \frac{1}{s_0 F(s_0/2)}$) so there exists $s_1 > |x|$ such that $\frac{4s_1}{\eta} F^{-1}\left(\frac{2s_1}{\eta t}\right) = 1$. Furthermore, using the fact that $t > \frac{1}{s_0 F(s_0/2)}$ we obtain

$$\frac{4s_1}{\eta} F^{-1}\left(\frac{2s_1}{\eta t}\right) = 1 = \frac{2}{s_0} F^{-1}\left(\frac{s_0 F(s_0/2)}{s_0}\right) \geq \frac{2}{s_0} F^{-1}\left(\frac{1}{s_0 t}\right) = \frac{4\frac{\eta}{2s_0}}{\eta} F^{-1}\left(\frac{2\frac{\eta}{2s_0}}{\eta t}\right)$$

which yields $\frac{\eta}{2s_1} < s_0$. We have then $\frac{2s_1}{\eta t} = F\left(\frac{\eta}{4s_1}\right) < F\left(\frac{\eta}{2s_1}\right) \leq \Psi\left(\frac{\eta}{2s_1}\right) \frac{2s_1}{\eta}$, hence $s_1 \leq \frac{1}{2}\eta h(t)$ and (13) in this case follows directly from (7), since $|x| < s_1$. \square

Proof of Theorem 1.1. It follows from (2) that for every $s_0 > 0$ we have

$$\Psi(s) \geq s^2 \int_{|y| < 1/s} |y|^2 \nu(dy) \geq c_1 s^2,$$

for $s < s_0$, where $c_1 = \int_{|y| < 1/s_0} |y|^2 \nu(dy) > 0$ since $\nu(\mathbb{R}^d) = \infty$. Therefore we have $\Psi(s) \geq sF(s)$ for $s < s_0$ and $F(s) = c_1 s$ and Lemma 3.2 yields

$$p_t(x) \geq c_2 h(t)^{-d} e^{-c_3 |x|^2/t},$$

for $t \in (\frac{2}{c_1 s_0^2}, \infty)$ and $|x| < c_4 s_0 t$. \square

In [30, Lemma 2] we obtained an upper estimate of densities for infinitely divisible distributions having Lévy measures with bounded support. Here we prove the opposite bound which holds also for more general class of processes.

LEMMA 3.3. Assume that (A1) holds and there exist $r_0 > 0$ and $\kappa_0 > 0$ such that

$$(15) \quad \inf_{x \in B(0, r_0)} \frac{d\nu_c}{dm}(x) = \kappa_0 > 0,$$

where $\frac{d\nu_c}{dm}$ denotes the Radon-Nikodym derivative of the absolutely continuous part ν_c of ν with respect to the Lebesgue measure. Then for every $\eta > 0$ there exists $c_1 = c_1(d), c_2 = c_2(d, \eta, M_0)$ such that

$$p_t(x) \geq c_1 r_0^{-d} \exp \left\{ -\frac{2|x|}{r_0} \log \left(\frac{c_2 |x|}{r_0^{d+1} \kappa_0 t} \right) \right\},$$

for $t \in (0, t_p)$, and $|x| \geq \max \left\{ r_0, \Psi \left(\frac{6\eta}{r_0} \right) \frac{3r_0}{4} t \right\}$.

Proof. Let

$$n = \left\lfloor \frac{4|x|}{3r_0} \right\rfloor + 1.$$

We note that $4|x|/(3r_0) < n \leq 7|x|/(3r_0)$, since $|x|/r_0 \geq 1$. Furthermore, $\frac{n}{t} \geq \Psi \left(\frac{6\eta}{r_0} \right) \frac{3nr_0}{4|x|} > \Psi \left(\frac{6\eta}{r_0} \right)$. It follows that $\Psi^{-1}(n/t) \geq \frac{6\eta}{r_0}$ and $\frac{r_0}{6} \geq \eta h(\frac{t}{n})$. From (15) and Proposition 2.1 for every $s \in (0, t_p)$ such that $r_0 > \eta h(s)$ we get

$$p_s(y) \geq c_1 s \kappa_0, \quad \eta h(s) \leq |y| < r_0,$$

for some constant $c_1 = c_1(d, \eta, M_0)$. Let $0 = x_0, x_1, \dots, x_{n-1}, x_n = x$ be such that $|x_{i+1} - x_i| = \frac{|x|}{n}$ (taking $x_i = (i/n)x$) and let $B_i = B(x_i, \frac{r_0}{8})$. We note that $\frac{3}{7}r_0 \leq \frac{|x|}{n} < \frac{3}{4}r_0$ and for $y_i \in B_i, y_{i+1} \in B_{i+1}$ we have $\eta h(\frac{t}{n}) \leq \frac{r_0}{6} < |y_{i+1} - y_i| < r_0$. We obtain

$$\begin{aligned} p_t(x) &= \int \dots \int p_{t/n}(y_1) p_{t/n}(y_2 - y_1) \dots p_{t/n}(x - y_{n-1}) dy_1 dy_2 \dots dy_{n-1} \\ &\geq \int_{B_1} \dots \int_{B_{n-1}} p_{t/n}(y_1) p_{t/n}(y_2 - y_1) \dots p_{t/n}(x - y_{n-1}) dy_1 dy_2 \dots dy_{n-1} \\ &\geq \left(c_1 \frac{t}{n} \kappa_0 \right)^n \left(\omega_d \left(\frac{r_0}{8} \right)^d \right)^{n-1} = \left(c_1 \frac{t}{n} \kappa_0 \omega_d \left(\frac{r_0}{8} \right)^d \right)^n \left(\omega_d \left(\frac{r_0}{8} \right)^d \right)^{-1} \\ &= c_2 r_0^{-d} \left(c_3 \frac{t}{n} \kappa_0 r_0^d \right)^n = c_2 r_0^{-d} \exp \left\{ -n \log \left(\frac{n}{c_3 t \kappa_0 r_0^d} \right) \right\} \\ &\geq c_2 r_0^{-d} \exp \left\{ -\frac{2|x|}{r_0} \log \left(\frac{2|x|}{c_3 t \kappa_0 r_0^{d+1}} \right) \right\}, \end{aligned}$$

with $c_2 = c_2(d), c_3 = c_3(d, \eta, M_0)$, and in the last line we use that $n \leq 2|x|/r_0$, since $\lfloor \frac{4}{3}u \rfloor + 1 \leq \frac{4}{3}u + \frac{2}{3}u = 2u$ for $u \geq \frac{3}{2}$, and $\lfloor \frac{4}{3}u \rfloor + 1 = 2 \leq 2u$ if $u \in [1, \frac{3}{2})$. \square

Proof of Theorem 1.2. It follows directly from Lemma 3.3. \square

4 Application to Lévy measures with bounded support

4.1 General case

Using above lemmas we obtain the following lower estimate of densities for semigroups with truncated Lévy measures.

THEOREM 4.1. *Assume that there exists constant $r_0 > 0$ such that $\text{supp } \nu \subset B(0, r_0)$ and*

$$(16) \quad \kappa_0 = \inf_{0 < |x| < r_0} \frac{d\nu_c}{dm}(x) > 0,$$

If (A1) holds with $t_p = \infty$ then there exist constants $c_i = c_i(d, M_0)$, $i = 1, 2, 3, 4$, such that

$$p_t(x) \geq c_1 \begin{cases} h(t)^{-d} & \text{for } |x| \leq \eta_0 h(t), t > 0, \\ t(h(t))^{-d} \nu(B(x, c_2 h(t))) & \text{for } \eta_0 h(t) \leq |x| \leq r_0, t \leq t_0, \\ h(t)^{-d} \exp \left\{ -\frac{c_3 |x|^2}{m_0 t} \right\}, & \text{for } \eta_0 h(t) \leq |x| \leq C_* t, t \geq t_0, \\ r_0^{-d} \exp \left\{ -\frac{2|x|}{r_0} \log \left(\frac{c_4 |x|}{r_0^{d+1} \kappa_0 t} \right) \right\}, & \text{for } |x| \geq r_0 \vee C_* t, t > 0, \end{cases}$$

where $m_0 = \int |y|^2 \nu(dy)$, $\eta_0 := \theta \wedge \frac{L_0}{216} \wedge 1$, $t_0 = \frac{4r_0^2}{\eta_0 L_0 m_0}$ and $C_* = \frac{\eta_0 L_0 m_0}{4r_0}$. We also have

$$(17) \quad \sqrt{L_0 m_0} \sqrt{t} \leq h(t) \leq \sqrt{2m_0} \sqrt{t} \quad \text{for } t > \frac{r_0^2}{L_0 m_0}.$$

Proof. The first estimate follows from (7) and the second from Proposition 2.1.

Using (2) we obtain

$$(18) \quad L_0 m_0 r^2 \leq \Psi(r) \leq 2m_0 r^2, \quad r \leq \frac{1}{r_0},$$

since $H(r) = r^2 \int |y|^2 \nu(dy) = m_0 r^2$, for $r \leq 1/r_0$. For $t > r_0^2/(L_0 m_0) \geq 1/\Psi(1/r_0)$ we have $h(t) \geq r_0$ and (17) follows by taking $r = 1/h(t)$ in (18).

Choosing $F(s) = L_0 m_0 s$ and $s_0 = 1/r_0$ in Lemma 3.2 we obtain

$$p_t(x) \geq c_1 h(t)^{-d} \exp \left\{ -\frac{2c_2 |x|^2}{\eta_0^2 L_0 m_0 t} \right\},$$

for $t > t_0 \geq \frac{2r_0^2}{L_0 m_0}$ and $|x| < \frac{\eta_0 L_0 m_0}{4r_0} t$. From (16) and Lemma 3.3 we get

$$p_t(x) \geq c_3 r_0^{-d} \exp \left\{ -\frac{2|x|}{r_0} \log \left(\frac{c_4 |x|}{r_0^{d+1} \kappa_0 t} \right) \right\},$$

for $|x| \geq \max \{r_0, C_* t\} \geq \max \left\{ r_0, \frac{54\eta_0^2 m_0}{r_0} t \right\}$. □

Now we deal with upper bounds. In the following lemma we improve the estimates obtained previously in [30, Lemma 2]. We use here in essential way the results of [21].

LEMMA 4.2. *Assume that the Lévy measure ν is symmetric and P_t has a transition density p_t for all $t > 0$. If, for some $r_0 > 0$, we have $\text{supp } \nu \subset B(0, r_0)$ then*

$$(19) \quad p_t(x) \leq e^{\frac{-|x|}{4r_0} \log\left(\frac{r_0|x|}{2tm_0}\right)} p_t(0), \quad |x| \geq \frac{2em_0}{r_0} t,$$

where $m_0 = \int |y|^2 \nu(dy)$. If additionally there exist constants $M_5, M_6 > 0$ such that

$$(20) \quad \int |y|^2 e^{|\xi||y|} \nu(dy) \leq M_5, \quad |\xi| \leq M_6,$$

then

$$p_t(x) \leq e^{-\frac{|x|^2}{4tM_5}} p_t(0), \quad |x| \leq 2M_5M_6t.$$

Proof. We use Theorem 6 of [21] obtaining

$$p_t(x) \leq e^{-D_t^2(x)} p_t(0), \quad x \in \mathbb{R}^d, t > 0,$$

where

$$D_t^2(x) = -v_t(\xi_0, x),$$

$$v_t(\xi, x) = -\xi \cdot x + t \int (\cosh(\xi \cdot y) - 1) \nu(dy),$$

and $\xi_0 = \xi_0(t, x) \in \mathbb{R}^d$ is such that $v_t(\xi_0, x) = \inf_{\xi \in \mathbb{R}^d} v_t(\xi, x)$. We have $\cosh(s) - 1 \leq s^2 e^s$ for all $s > 0$, therefore

$$v_t(\xi, x) \leq -\xi \cdot x + t|\xi|^2 \int |y|^2 e^{|\xi||y|} \nu(dy) \leq -\xi \cdot x + t|\xi|^2 e^{|\xi|r_0} m_0.$$

We choose $s > 0$ such that $se^{sr_0} = \frac{|x|}{2tm_0}$. If $\frac{|x|}{2tm_0} \geq \frac{e}{r_0}$ then $\frac{1}{2r_0} \log\left(\frac{r_0|x|}{2tm_0}\right) \leq s \leq \frac{1}{r_0} \log\left(\frac{r_0|x|}{2tm_0}\right)$, since $e^u \leq ue^u \leq e^{2u}$ for $u \geq 1$. Taking $\xi_1 = \frac{sx}{|x|}$ we obtain

$$v_t(\xi_0, x) \leq v_t(\xi_1, x) \leq -\frac{1}{2}s|x| \leq \frac{-|x|}{4r_0} \log\left(\frac{r_0|x|}{2tm_0}\right),$$

and (19) follows. If (20) is satisfied then

$$v_t(\xi, x) \leq -\xi \cdot x + t|\xi|^2 \int |y|^2 e^{|\xi||y|} \nu(dy) \leq -\xi \cdot x + t|\xi|^2 M_5,$$

for $|\xi| \leq M_6$. Taking $\xi_2 = \frac{1}{2tM_5}x$ we obtain

$$v_t(\xi_0, x) \leq v_t(\xi_2, x) \leq \frac{-|x|^2}{4tM_5},$$

for $|x| \leq 2M_5M_6t$. □

We summarize estimates obtained in Lemma 4.2, Proposition 2.2 and (7) in the following Theorem. We recall that the first estimate holds in fact for every process satisfying (A1). We note also that for $t > 1$ we have $h(t) \asymp \sqrt{t}$ and the exponential term in the second inequality below dominates the forth bound so for $|x| > C^*t$ the latter estimate is more exact. Similarly the third estimate is more exact then the second for $|x| < C^*t$ and $t > 1$. For small t the result of analogous comparison depends on functions h and f . We will compare the bounds more precisely in the next section under additional assumptions on ν .

THEOREM 4.3. *Assume that (A1) holds with $t_p = \infty$, $\text{supp}(\nu) \subset B(0, r_0)$ for some $r_0 > 0$ and there exist a constant γ and a nonincreasing function f such that (9) and (10) hold. Then there exist constants $\theta = \theta(d, M_0)$, $c_1 = c_1(d, M_0, M_3, M_4)$, $c_2 = c_2(d, M_0)$, $c_3 = c_3(d, M_0)$ such that*

$$p_t(x) \leq c_1 \begin{cases} h(t)^{-d} & \text{for } |x| \leq \theta h(t), \\ t [h(t)]^{\gamma-d} f(|x|/4) + h(t)^{-d} e^{-c_2 \frac{|x|}{h(t)} \log\left(1 + \frac{c_3|x|}{h(t)}\right)} & \text{for } |x| \geq \theta h(t) \\ h(t)^{-d} \exp\left\{-\frac{|x|^2}{4em_0 t}\right\}, & \text{for } |x| \leq C^*t, \\ h(t)^{-d} \exp\left\{-\frac{|x|}{4r_0} \log\left(\frac{r_0|x|}{2m_0 t}\right)\right\}, & \text{for } |x| \geq C^*t, \end{cases}$$

where $C^* = \frac{2em_0}{r_0}$, and $m_0 = \int |y|^2 \nu(dy)$.

Proof. The first inequality follows from (7) and the second from Proposition 2.2. It follows from Lemma 4.2 and (7) that

$$p_t(x) \leq c_1 h(t)^{-d} e^{-\frac{|x|}{4r_0} \log\left(\frac{r_0|x|}{2tm_0}\right)}, \quad |x| \geq \frac{2em_0}{r_0} t.$$

Taking $M_6 = \frac{1}{r_0}$ and $M_5 = em_0$ in (20) we get

$$p_t(x) \leq c_1 h(t)^{-d} e^{-\frac{|x|^2}{4tem_0}}, \quad |x| \leq \frac{2em_0}{r_0} t.$$

□

4.2 Absolutely continuous Lévy measures

In this section we always assume that the Lévy measure ν is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d \setminus \{0\}$ with a density $\bar{\nu}$. Moreover we assume that there exist constants $m_1, m_2, r_0 > 0$ and a nonincreasing function $f : (0, r_0] \rightarrow (0, \infty)$ such that

$$(21) \quad m_1 f(|x|) \leq \bar{\nu}(x) \leq m_2 f(|x|), \quad \text{for } |x| < r_0.$$

Apart from Lemma 4.4 we consider here ν such that $\text{supp}(\nu) \subset B(0, r_0)$.

We will also assume that f satisfies **(A2)**. We note that for a nonincreasing functions **(A2)** yields the following both side doubling property

$$(22) \quad c_1 f(r) \leq f(2r) \leq c_2 f(r), \quad 2r \leq r_0,$$

for some constants $c_1, c_2 > 0$. If f is nonincreasing and (22) holds with $2^{-d-2} < c_1 \leq c_2 < 2^{-d}$ then **(A2)** is satisfied.

The condition **(A2)** holds for many typical functions f such as $f(s) = s^{-d-\alpha}$, $\alpha \in (0, 2)$, or $f(s) = s^{-d-\alpha}(\log(1 + \frac{1}{s}))^{-\beta}$, $\beta \in \mathbb{R}$. However we note that a density $\bar{\nu}$ of a Lévy measure not always satisfies the doubling property at the origin. For example we can observe it for

$$\bar{\nu}(x) = \frac{2^{(2+d)k^2}}{k^2 + 1}, \quad \text{for } 2^{-(k+1)^2} < |x| \leq 2^{-k^2},$$

since

$$\frac{f(2 \cdot 2^{-k^2})}{f(2^{-k^2})} = \frac{k^2 + 1}{(k-1)^2 + 1} 2^{(2+d)(-2k+1)} \rightarrow 0, \quad \text{as } k \rightarrow \infty.$$

The constants appearing in this section can all depend on $m_1, m_2, \beta_1, \beta_2, M_1, M_2, r_0, f$ and ν and we will not mention it explicitly below. We will use the notation $f \asymp g$ to indicate that there exist constant c_1, c_2 such that $c_1 g \leq f \leq c_2 g$. In the following lemmas we obtain some interesting properties of semigroups satisfying above conditions.

LEMMA 4.4. *Assume that the Lévy measure ν satisfies (21) and the function f satisfies **(A2)**. If $\kappa = \inf_{|x| \leq r_0} f(x) > 0$ then*

$$(23) \quad \bar{\nu}(x) \asymp \frac{\Psi(1/|x|)}{|x|^d}, \quad |x| < r_0.$$

Proof. From (2) we get

$$\Psi(1/r) \asymp r^{-2} \int_{|y| < r} |y|^2 \bar{\nu}(y) dy + \int_{|y| \geq r} \bar{\nu}(y) dy.$$

We observe that

$$\int_{|y| < r} |y|^2 \bar{\nu}(y) dy \geq m_1 f(r) \int_{|y| < r} |y|^2 dy = \frac{m_1 d \omega_d}{d+2} r^{d+2} f(r), \quad r < r_0,$$

where ω_d denotes the volume of the unit ball in \mathbb{R}^d , and the lower estimate in (23) follows.

Now we prove the upper bound. For $r \leq r_0$, using (21) and **(A2)**, we obtain

$$\int_{|y| < r} |y|^2 \bar{\nu}(y) dy \leq \int_{|y| < r} |y|^2 m_2 M_2 \left(\frac{r}{|y|} \right)^{\beta_2} f(r) dy = \frac{d \omega_d m_2 M_2}{2 + d - \beta_2} r^{d+2} f(r).$$

Furthermore

$$\begin{aligned} \int_{r \leq |y| \leq r_0} \bar{\nu}(y) dy &\leq m_2 d \omega_d \int_r^{r_0} s^{d-1} f(s) ds \leq \frac{m_2 d \omega_d r^{\beta_1} f(r)}{M_1} \int_r^{r_0} s^{d-1-\beta_1} ds \\ &\leq c_1 r^d f(r), \end{aligned}$$

and from **(A2)** we obtain $r^d f(r) \geq M_1 r_0^{\beta_1} r^{d-\beta_1} f(r_0) \geq M_1 r_0^d \kappa$, for $r < r_0$, and this yields

$$\int_{|y| \geq r} \bar{\nu}(y) dy \leq c_1 r^d f(r) + \int_{r_0 < |y|} \bar{\nu}(y) dy \leq \left(c_1 + \frac{\int_{r_0 < |y|} \bar{\nu}(y) dy}{M_1 r_0^d \kappa} \right) r^d f(r),$$

and the lemma follows. \square

LEMMA 4.5. Assume that ν satisfies (21) and **(A2)** and $\text{supp } \nu \subset B(0, r_0)$. If $\kappa = \inf_{|x| \leq r_0} f(x) > 0$ then **(A1)** holds with $t_p = \infty$.

Proof. First we will prove that $\Psi(|\xi|) \asymp \Phi(\xi)$. The inequality $\Phi(\xi) \leq \Psi(|\xi|)$ follows directly from the definition of Ψ so we have to prove only the opposite estimate. Using Lemma 4.4 for $|\xi| > \frac{1}{r_0}$ we get

$$\begin{aligned} \Phi(\xi) &= \int (1 - \cos(\xi \cdot y)) \bar{\nu}(y) dy \\ &\geq m_1 \int (1 - \cos(\xi \cdot y)) f(|y|) dy \\ &\geq c_1 \int_{|y| < \frac{1}{|\xi|}} (1 - \cos(\xi \cdot y)) \frac{\Psi\left(\frac{1}{|y|}\right)}{|y|^d} dy \\ &\geq c_2 \Psi(|\xi|) \int_{|y| < \frac{1}{|\xi|}} (\xi \cdot y)^2 \frac{1}{|y|^d} dy \\ &= c_3 \Psi(|\xi|), \end{aligned}$$

since $\int_{|y| < \frac{1}{|\xi|}} (\xi \cdot y)^2 \frac{1}{|y|^d} dy = |\xi|^2 \int_{|y| < \frac{1}{|\xi|}} \left(\frac{\xi}{|\xi|} \cdot y \right)^2 \frac{1}{|y|^d} dy = |\xi|^2 \int_{|y| < \frac{1}{|\xi|}} \frac{y_1^2}{|y|^d} dy = \text{const.}$, where we use the rotational invariance of the Lebesgue measure. This and Lemma 4.4 yield

$$(24) \quad \Phi(\xi) \asymp \Psi(|\xi|) \asymp f(1/|\xi|) |\xi|^{-d}, \quad |\xi| > \frac{1}{r_0}.$$

For $|\xi| \leq \frac{1}{r_0}$ we have

$$\Phi(\xi) = \int (1 - \cos(\xi \cdot y)) \bar{\nu}(y) dy \geq c_4 \int_{|y| < r_0} |\xi \cdot y|^2 f(|y|) dy \geq c_5 |\xi|^2.$$

Further, by (2) we have

$$\Psi(r) \asymp r^2 \int_{|y| < r_0} |y|^2 \nu(dy), \quad r \leq \frac{1}{r_0},$$

and so

$$(25) \quad \Phi(\xi) \asymp \Psi(|\xi|) \asymp |\xi|^2, \quad |\xi| \leq \frac{1}{r_0}.$$

It follows from (24),(25) and **(A2)** that there exist $L > 1$ and $c_* > 1$ such that

$$\Psi(Lr) \geq c_* \Psi(r), \quad r > 0.$$

This yields

$$\Psi(L^n/h(t)) \geq c_*^n \Psi(1/h(t)) = \frac{c_*^n}{t}, \quad t > 0, n \in \mathbb{N},$$

and we obtain

$$\begin{aligned} \int_{\mathbb{R}^d} e^{-t \operatorname{Re}(\Phi(\xi))} |\xi| d\xi &\leq \int e^{-c_6 t \Psi(|\xi|)} |\xi| d\xi \\ &= \int_{|\xi| \leq 1/h(t)} e^{-c_6 t \Psi(|\xi|)} |\xi| d\xi + \int_{|\xi| > 1/h(t)} e^{-c_6 t \Psi(|\xi|)} |\xi| d\xi \\ &\leq c_7 h(t)^{-d-1} + \sum_{n=0}^{\infty} \int_{\frac{L^n}{h(t)} < |\xi| \leq \frac{L^{n+1}}{h(t)}} e^{-c_6 t \Psi(|\xi|)} |\xi| d\xi \\ &\leq c_7 h(t)^{-d-1} + c_8 \sum_{n=0}^{\infty} e^{-c_6 t \Psi(L^n/h(t))} \left(\frac{L^{n+1}}{h(t)} \right)^{d+1} \\ &\leq c_7 h(t)^{-d-1} + c_9 h(t)^{-d-1} \sum_{n=0}^{\infty} e^{-c_6 c_*^n L^{(n+1)(d+1)}} \\ &= c_{10} h(t)^{-d-1}, \end{aligned}$$

and the lemma follows. \square

Using the above properties we can improve now the estimates obtained previously in Theorem 4.3 in the general case.

THEOREM 4.6. *Assume that $\operatorname{supp}(\nu) \subset B(0, r_0)$, ν is symmetric and absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d \setminus \{0\}$ with a density $\bar{\nu}$ and there exists a nonincreasing function $f : [0, r_0] \rightarrow [0, \infty]$ such that*

$$m_1 f(|x|) \leq \bar{\nu}(x) \leq m_2 f(|x|), \quad 0 < |x| < r_0.$$

where f satisfies **(A2)**, and $\kappa = \inf_{s \in (0, r_0]} f(s) > 0$. Then there exist $c_1, c_2, c_3, c_4, \eta_1$ and C^* such that

$$(26) \quad p_t(x) \leq c_1 \begin{cases} h(t)^{-d} & \text{for } |x| \leq \eta_1 h(t), t > 0, \\ tf(|x|) & \text{for } \eta_1 h(t) \leq |x| \leq r_0, t \leq t_1, \\ h(t)^{-d} \exp \left\{ -\frac{c_2 |x|^2}{t} \right\}, & \text{for } \eta_1 h(t) \leq |x| \leq C^* t, t \geq t_1, \\ \exp \left\{ -c_3 |x| \log \left(\frac{c_4 |x|}{t} \right) \right\}, & \text{for } |x| \geq r_0 \vee C^* t, t > 0, \end{cases}$$

where $t_1 = r_0/C^*$.

Proof. First we prove the second inequality. Let $f_*(s) = f(s)$ for $s < r_0$ and $f_*(s) = \kappa$ for $s \geq r_0$. We have $\nu(A) \leq c_1 f_*(\delta(A)) (\text{diam}(A))^d$ for every Borel set A . It follows from **(A2)** that $f_* \left(s \vee |y| - \frac{|y|}{2} \right) \leq f_* \left(\frac{s}{2} \right) \leq c_2 f_*(s)$, for all $y \in \mathbb{R}^d$, $s > 0$, and (10) holds for f_* since $\nu(B(0, r)^c) \leq c_3 \Psi(1/r)$ by (2). Lemma 4.5 yields that **(A1)** holds and from Proposition 2.2 we obtain

$$(27) \quad p_t(x) \leq c_4 (h(t))^{-d} \min \left\{ 1, t [h(t)]^d f_* (|x|/4) + e^{-c_5 \frac{|x|}{h(t)} \log \left(1 + \frac{c_6 |x|}{h(t)} \right)} \right\},$$

for all $x \in \mathbb{R}^d \setminus \{0\}$ and $t > 0$. Now we will show that

$$t [h(t)]^d f_* (|x|/4) \geq c e^{-c_5 \frac{|x|}{h(t)} \log \left(1 + \frac{c_6 |x|}{h(t)} \right)},$$

for $|x| > \eta_1 h(t)$, and hence the exponential term in (27) can be omitted. We observe that (24) and (25) yield

$$t = \frac{1}{\Psi(1/h(t))} \asymp \begin{cases} h(t)^2 & \text{for } t \geq \frac{1}{\Psi(1/r_0)}, \\ \frac{1}{f(h(t))h(t)^d} & \text{for } t < \frac{1}{\Psi(1/r_0)}. \end{cases}$$

Therefore, for $|x| > 4r_0$ and $t \geq \frac{1}{\Psi(1/r_0)}$ we have $t [h(t)]^d f_* (|x|/4) = t [h(t)]^d \kappa \geq (|x|/h(t))^{-d-2} |x|^{d+2} \geq c_7 e^{-c_5 \frac{|x|}{h(t)} \log \left(1 + \frac{c_6 |x|}{h(t)} \right)}$, and for $|x| > 4r_0$ and $t \leq \frac{1}{\Psi(1/r_0)}$ by **(A2)** we have $t [h(t)]^d f_* (|x|/4) = t [h(t)]^d \kappa \geq c_8 / f(h(t)) \geq c_9 (h(t))^{\beta_2} \geq c_{10} e^{-c_5 \frac{4r_0}{h(t)} \log \left(1 + \frac{c_6 4r_0}{h(t)} \right)} \geq c_{10} e^{-c_5 \frac{|x|}{h(t)} \log \left(1 + \frac{c_6 |x|}{h(t)} \right)}$. For $|x| \leq 4r_0$ and a constant $\eta_1 < 4$ from Lemma 4.4 we get

$$(28) \quad f(|x|/4) \asymp \frac{\Psi(4/|x|)}{|x|^d} \geq \frac{\Psi(\eta_1/|x|)}{|x|^d}.$$

We note that (2) yields

$$(29) \quad \Psi(2r) \leq 2H(2r) \leq 8H(r) \leq \frac{8}{L_0} \Psi(r), \quad r > 0.$$

For $\eta_1 h(t) < |x| \leq 4r_0$ from (28) and (29) we obtain

$$\begin{aligned} th(t)^d f(|x|/4) &= \frac{h(t)^d f(|x|/4)}{\Psi(1/h(t))} \geq c_{11} \frac{h(t)^d \Psi(\eta_1/|x|)}{\Psi(1/h(t))|x|^d} \\ &\geq \frac{c_{11} L_0}{8\eta_1^d} \left(\frac{|x|}{\eta_1 h(t)} \right)^{-\log_2 \frac{8}{L_0} - d} \\ &\geq c_{12} e^{-c_5 \frac{|x|}{h(t)} \log\left(1 + \frac{c_6 |x|}{h(t)}\right)}, \end{aligned}$$

hence (27) and **(A2)** yield

$$(30) \quad p_t(x) \leq c_{13} t f_*(|x|/4) \leq c_{14} t f_*(|x|), \quad |x| > \eta_1 h(t),$$

for every $\eta_1 \in (0, 4)$ and $c_{14} = c_{14}(\eta_1)$, which gives in particular the second case in (26).

Now we will prove the last inequality. Lemma 4.2 yields

$$p_t(x) \leq c_{15} h(t)^{-d} e^{\frac{-|x|}{4r_0} \log\left(\frac{r_0 |x|}{2tm_0}\right)}, \quad |x| \geq \frac{2em_0}{r_0} t.$$

From Lemma 4.4 we have $\Psi(1/r) \asymp r^d f(r)$, and since $r^{\beta_1} f(r) \geq M_1 r_0^{\beta_1} f(r_0) \geq M_1 r_0^{\beta_1} \kappa$, we get $\Psi(1/r) > c_{16} r^{d-\beta_1}$ for $r \leq r_0$. This yields $\frac{1}{r} \geq \Psi^{-1}(c_{16} r^{d-\beta_1})$ and $h(r^{\beta_1-d}/c_{16}) \geq r$, hence we obtain $h(t) \geq (c_{16} t)^{1/(\beta_1-d)}$ for $t < r_0^{\beta_1-d}/c_{16}$. For $|x| > \frac{2m_0}{c_{16} r_0}$ we get

$$h(t) \geq \left(\frac{2tm_0}{r_0 |x|} \right)^{1/(\beta_1-d)},$$

and this yields

$$\begin{aligned} h(t)^{-d} \exp \left\{ -\frac{|x|}{4r_0} \log \left(\frac{r_0 |x|}{2tm_0} \right) \right\} &= h(t)^{-d} \left(\frac{2tm_0}{r_0 |x|} \right)^{\frac{|x|}{4r_0}} \\ &\leq \left(\frac{2tm_0}{r_0 |x|} \right)^{\frac{|x|}{4r_0} - \frac{d}{\beta_1-d}} \leq \left(\frac{2tm_0}{r_0 |x|} \right)^{\frac{|x|}{8r_0}} \\ &= \exp \left\{ -\frac{|x|}{8r_0} \log \left(\frac{r_0 |x|}{2tm_0} \right) \right\}, \end{aligned}$$

provided $|x| \geq R_0 = \max\left\{\frac{8r_0 d}{\beta_1-d}, \frac{2r_0^{\beta_1-d} m_0}{c_{16} r_0}, \frac{2m_0}{c_{16} r_0}\right\}$ and $t < r_0^{\beta_1-d}/c_{16}$. For $t \geq r_0^{\beta_1-d}/c_{16}$ we have $h(t)^{-d} \leq h(r_0^{\beta_1-d}/c_{16})^{-d} \leq r_0^{-d}$. We obtain

$$(31) \quad p_t(x) \leq c_{17} e^{\frac{-|x|}{8r_0} \log\left(\frac{r_0 |x|}{2tm_0}\right)}, \quad |x| \geq R_0 \vee \frac{2em_0}{r_0} t, \quad t > 0.$$

For $r_0 \vee \left(\frac{2em_0}{r_0} t\right) \leq |x| \leq R_0$ we observe that

$$c_{14} t f_*(|x|) = c_{14} \kappa t \leq c_{18} e^{\frac{-|x|}{R_0} \log\left(\frac{r_0 |x|}{2tm_0}\right)},$$

where $c_{18} = \frac{c_{14}\kappa r_0}{2m_0} R_0$, and the last inequality in (26) with $C^* = \frac{2em_0}{r_0}$ follows from this, (31) and (30) since $\eta_1 h(t) \leq r_0$, for $t \leq \frac{r_0 R_0}{2em_0}$ and $\eta_1 = \theta \wedge \sqrt{\frac{er_0}{R_0}} \wedge 1$ (note that $\eta_1 h(t) \leq r_0$, provided $t \leq \frac{1}{\Psi(\eta_1/r_0)}$, and that $\Psi(\eta_1/r_0) \leq 2m_0\eta_1^2/r_0^2$ by (2), since $\eta_1 \leq 1$).

Taking $M_6 = \frac{1}{r_0}$ and $M_5 = em_0$ in (20) we get

$$p_t(x) \leq c_{19} h(t)^{-d} e^{-\frac{|x|^2}{4tem_0}}, \quad |x| \leq \frac{2em_0}{r_0} t,$$

which gives the third inequality in (30). The first inequality in (30) follows from (7), since $\eta_1 \leq \theta$. \square

We can prove now Theorem 1.3.

Proof of Theorem 1.3. We use here the constants C_*, t_0, η_0 from Theorem 4.1. We note that $C^* > C_*$ and $t_1 < t_0$ since $\eta_0 \leq 1$ and $L_0 \leq 2$. Let

$$\eta_* = \eta_0 \vee \eta_1.$$

We obtain the first estimate using (7) since $\eta_0 \vee \eta_1 < \theta$. The inequalities in 2. and 4. follows directly from Theorem 4.1 and Theorem 4.6. Similarly, the third estimate for $\eta_* h(t) \leq |x| \leq C_* t$, $t \geq t_0$ is a direct consequence of these theorems. For $r_0 \vee C_* t \leq |x| \leq C^* t$ and $t \geq t_1$ we have

$$p_t(x) \geq c_1 e^{-c_2 |x| \log(\frac{c_3 |x|}{t})} \geq c_1 e^{-c_2 |x| \log(c_3 C^*)} \geq c_1 h(t_1)^d h(t)^{-d} e^{-c_2 C_*^{-1} \log(c_3 C^*) \frac{|x|^2}{t}},$$

and for $t_1 \leq t \leq t_0$ and $\eta_* h(t) \leq |x| \leq r_0$ we have

$$tf(|x|) \asymp h(t)^{-d} e^{-\frac{c_4 |x|^2}{t}} \asymp \text{const.}$$

and the inequalities in 4. follow. \square

5 Application to tempered stable processes

Let

$$\nu(A) \asymp \int_0^\infty \int_{\mathbb{S}} \mathbb{1}_A(s\theta) s^{-1-\alpha} (1+s)^\kappa e^{-ms^\beta} ds \mu(d\theta),$$

where μ is bounded, symmetric and nondegenerate measure on the unit sphere \mathbb{S} , $m > 0$, $\beta \in (0, 1]$, $\alpha \in (0, 2)$, $\kappa \in (-\infty, 1 + \alpha]$.

We have here

$$(32) \quad \Phi(\xi) \asymp \Psi(|\xi|) \asymp |\xi|^2 \wedge |\xi|^\alpha,$$

which follows from Proposition 1 and Corollary 2 in [17]. We get

$$(33) \quad h(t) \asymp t^{1/2} \wedge t^{1/\alpha},$$

and Lemma 5 in [17] yields that **(A1)** is satisfied with $t_p = \infty$.

Such examples were discussed previously in [30],[17] and [18]. It was proved (see also Proposition 2.2 above) that if μ is a $\gamma - 1$ - measure on \mathbb{S} , i.e., there exists a constants c such that

$$\mu(B(\theta, \rho) \cap \mathbb{S}) \leq c\rho^{\gamma-1}, \quad \theta \in \mathbb{S},$$

and that there exist $D_0 \subset \mathbb{S}$ and $c > 0$ such that

$$\mu(B(\theta, \rho) \cap \mathbb{S}) \geq c\rho^{\gamma-1}, \quad \theta \in D_0,$$

for some $\gamma \in [1, d]$, then

$$\begin{aligned} p_t(x) &\leq c_1 t^{-d/\alpha} \min \left\{ 1, t^{1+\gamma/\alpha} |x|^{-\gamma-\alpha} (1+|x|)^\kappa e^{-m|x|^\beta/4^\beta} + e^{-c_2 t^{-1/\alpha} |x| \log(1+c_3 t^{-1/\alpha} |x|)} \right\}, \\ &\leq c_4 t^{-d/\alpha} \min \left\{ 1, t^{1+\gamma/\alpha} |x|^{-\gamma-\alpha} (1+|x|)^\kappa e^{-m|x|^\beta/4^\beta} \right\}, \\ &x \in \mathbb{R}^d, t \in (0, 1], \end{aligned}$$

and

$$\begin{aligned} p_t(x) &\leq c_4 t^{-d/2} \min \left\{ 1, t^{1+\gamma/2} |x|^{-\gamma-\alpha} (1+|x|)^\kappa e^{-m|x|^\beta/4^\beta} + e^{-c_5 t^{-1/2} |x| \log(1+c_6 t^{-1/2} |x|)} \right\}, \\ &x \in \mathbb{R}^d, t \in (1, \infty). \end{aligned}$$

Note that we can omit the exponential term in the first estimate since there exists $c > 0$ such that $s^{-\gamma-\alpha} (1+s)^\kappa e^{-ms^\beta/4^\beta} \geq ce^{-c_2 s \log(1+c_3 s)}$ for $s > 0$ and for $t < 1$ we have $t^{1+\gamma/\alpha} |x|^{-\gamma-\alpha} (1+|x|)^\kappa e^{-m|x|^\beta/4^\beta} \geq c(t^{-1/\alpha} |x|)^{-\gamma-\alpha} (1+t^{-1/\alpha} |x|)^\kappa e^{-m(t^{-1/\alpha} |x|)^\beta/4^\beta}$. Similar procedure for large times is not possible.

More precise estimates for small t were obtained in [18]. If $(\beta, \kappa) \in (0, 1) \times (-\infty, 1 + \alpha]$ or $(\beta, \kappa) \in \{1\} \times (-\infty, \alpha]$ then

$$p_t(x) \leq c_7 t^{1+\frac{\gamma-d}{\alpha}} |x|^{-\gamma-\alpha+\kappa} e^{-m|x|^\beta}, \quad t \in (0, 1], |x| \geq 4,$$

and

$$p_t(x) \geq c_8 t^{1+\frac{\gamma-d}{\alpha}} |x|^{-\gamma-\alpha+\kappa} e^{-m|x|^\beta}, \quad t \in (0, 1], x \in D,$$

where $D = \{x \in \mathbb{R}^d : x = r\theta, r \geq 4, \theta \in D_0\}$.

Here we improve the estimates for large values of t .

Proof of Theorem 1.4. We will need the following preparation. As usual (see [30, 17, 18]) we divide the Lévy measure in the two parts. For $r > 0$ we denote

$$\tilde{\nu}_r(dy) = \mathbb{1}_{B(0,r)}(y) \nu(dy) \quad \text{and} \quad \bar{\nu}_r(dy) = \mathbb{1}_{B(0,r)^c}(y) \nu(dy).$$

In terms of the corresponding Lévy process, $\tilde{\nu}_r$ is related to the jumps which are close to the origin, while $\bar{\nu}_r$ represents the large jumps.

For the restricted Lévy measures we consider the two semigroups of measures $\{\tilde{P}_t^r, t \geq 0\}$ and $\{\bar{P}_t^r, t \geq 0\}$ such that

$$\mathcal{F}(\tilde{P}_t^r)(\xi) = \exp \left(t \int (e^{i\xi \cdot y} - 1 - i\xi \cdot y) \tilde{\nu}_r(dy) \right), \quad \xi \in \mathbb{R}^d,$$

and

$$\mathcal{F}(\bar{P}_t^r)(\xi) = \exp \left(t \int (e^{i\xi \cdot y} - 1) \bar{\nu}_r(dy) \right), \quad \xi \in \mathbb{R}^d,$$

respectively. We have

$$\begin{aligned} |\mathcal{F}(\tilde{P}_t^r)(\xi)| &= \exp \left(-t \int_{|y| < r} (1 - \cos(y \cdot \xi)) \nu(dy) \right) \\ &= \exp \left(-t \left(\operatorname{Re}(\Phi(\xi)) - \int_{|y| \geq r} (1 - \cos(y \cdot \xi)) \nu(dy) \right) \right) \\ (34) \quad &\leq \exp(-t \operatorname{Re}(\Phi(\xi))) \exp(2t\nu(B(0, r)^c)), \quad \xi \in \mathbb{R}^d, \end{aligned}$$

and, therefore, by (32), for every $r > 0$ and $t > 0$ the measures \tilde{P}_t^r are absolutely continuous with respect to the Lebesgue measure with densities $\tilde{p}_t^r \in C_b^1(\mathbb{R}^d)$.

We have

$$P_t = \tilde{P}_t^r * \bar{P}_t^r, \quad \text{and} \quad p_t = \tilde{p}_t^r * \bar{P}_t^r, \quad t > 0,$$

where

$$\begin{aligned} (35) \quad \bar{P}_t^r &= \exp(t(\bar{\nu}_r - |\bar{\nu}_r| \delta_0)) = \sum_{n=0}^{\infty} \frac{t^n (\bar{\nu}_r - |\bar{\nu}_r| \delta_0)^{n*}}{n!} \\ &= e^{-t|\bar{\nu}_r|} \sum_{n=0}^{\infty} \frac{t^n \bar{\nu}_r^{n*}}{n!}, \quad t \geq 0. \end{aligned}$$

We will estimate first the densities $\tilde{p}_t^r \in C_b^1(\mathbb{R}^d)$ using Lemma 4.2 and Theorem 6 of [21]. Let

$$\nu(A) \leq c_1 \int_0^\infty \int_{\mathbb{S}} \mathbb{1}_A(s\theta) s^{-1-\alpha} (1+s)^\kappa e^{-ms^\beta} ds \mu(d\theta),$$

and $L = c_1 |\mu| \int_0^\infty s^{1-\alpha} (1+s)^\kappa e^{-(1-2^{-\beta/2})ms^\beta} ds$. Using Lemma 4.2 with $M_5 = L$, $M_6 = \frac{1}{2^{\beta/2}} m r^{\beta-1}$ we get

$$(36) \quad \tilde{p}_t^r(x) \leq \tilde{p}_t^r(0) e^{-\frac{|x|^2}{4tL}}, \quad |x| \leq 2^{1-\beta/2} L m r^{\beta-1} t.$$

Recall that Theorem 6 of [21] yields

$$\tilde{p}_t^r(x) \leq e^{-D_t^2(x)} \tilde{p}_t^r(0), \quad x \in \mathbb{R}^d, t > 0,$$

where $D_t^2(x) = -v_t(\xi_0, x)$, $v_t(\xi, x) = -\xi \cdot x + t \int (\cosh(\xi \cdot y) - 1) \tilde{\nu}_r(dy)$, and $\xi_0 = \xi_0(t, x) \in \mathbb{R}^d$ is such that $v_t(\xi_0, x) = \inf_{\xi \in \mathbb{R}^d} v_t(\xi, x)$.

If $|x| > r$ then

$$\begin{aligned} \int |y|^2 e^{|\xi||y|} \tilde{\nu}_r(dy) &\leq \int_{|y| < |x|} |y|^2 e^{|\xi||y|} \nu(dy) \\ &\leq c_1 |\mu| \int_0^{|x|} s^{1-\alpha} (1+s)^\kappa e^{-(1-2^{-\beta/2})ms^\beta} ds \leq L, \end{aligned}$$

provided $|\xi| \leq \frac{1}{2^{\beta/2}m} |x|^{\beta-1}$.

Let $R_0 = \frac{2^{\beta/2}(1-2^{-\beta/2})}{mL}$. Taking $\xi_1 = \frac{1}{2^{\beta/2}m} |x|^{\beta-2} x$ for $|x| \geq (t/R_0)^{\frac{1}{2-\beta}}$ we obtain

$$\begin{aligned} v_t(\xi_1, x) &\leq -\frac{1}{2^{\beta/2}m} |x|^\beta + t \frac{1}{2^\beta} m^2 |x|^{2(\beta-1)} L \leq \frac{1}{2^{\beta/2}m} |x|^\beta \left(-1 + \frac{1}{2^{\beta/2}m} t m |x|^{\beta-2} L \right) \\ &\leq -\frac{1}{2^\beta} m |x|^\beta. \end{aligned}$$

This yields

$$(37) \quad \tilde{p}_t^r(x) \leq \tilde{p}_t^r(0) e^{-\frac{1}{2^\beta} m |x|^\beta}, \quad |x| \geq \max\{r, (R_0)^{\frac{1}{\beta-2}} t^{\frac{1}{2-\beta}}\}.$$

As usual, below we will use \tilde{P}_t^r , \tilde{p}_t^r and \bar{P}_t^r with $r = h(t)$ and for simplification we will write $\tilde{P}_t = \tilde{P}_t^{h(t)}$, $\tilde{p}_t = \tilde{p}_t^{h(t)}$ and $\bar{P}_t = \bar{P}_t^{h(t)}$.

For t large enough, by (36), (37) and (33) we have

$$\tilde{p}_t(x) \leq \tilde{p}_t(0) e^{-\frac{|x|^2}{4tL}}, \quad |x| \leq c_2 t^{\frac{\beta+1}{2}},$$

and

$$\tilde{p}_t(x) \leq \tilde{p}_t(0) e^{-\frac{1}{2^\beta} m |x|^\beta}, \quad |x| \geq c_3 t^{\frac{1}{2-\beta}},$$

and since $\tilde{p}_t(0) \leq c_4 h(t)^{-d}$ (see Lemma 8 in [17]), and for sufficiently large t we have $c_3 t^{\frac{1}{2-\beta}} \leq c_2 t^{\frac{\beta+1}{2}}$, we obtain

$$(38) \quad \tilde{p}_t(x) \leq \tilde{p}_t(0) e^{-\left(\frac{|x|^2}{4tL} \wedge \frac{m|x|^\beta}{2^\beta}\right)} \leq c_4 h(t)^{-d} e^{-\left(\frac{|x|^2}{4tL} \wedge \frac{m|x|^\beta}{2^\beta}\right)}, \quad x \in \mathbb{R}^d.$$

We have $\Psi(1/h(t)) = 1/t$ and it follows from Corollary 10 in [17] with $\gamma = 1$ and (35) that

$$(39) \quad \bar{P}_t(B(x, \rho)) \leq c_5 t f(|x|/4) \rho,$$

for $\rho \leq \frac{1}{2}|x|$ and $t > 0$, where

$$f(s) = s^{-1-\alpha} (1+s)^\kappa e^{-ms^\beta}, \quad s > 0.$$

We fix t and denote

$$g(s) = e^{-\left(\frac{s^2}{4tL} \wedge \frac{ms^\beta}{2^\beta}\right)}, \quad s \geq 0.$$

We note that g is decreasing and continuous on $[0, \infty)$, and the inverse function is given by

$$g^{-1}(s) = \sqrt{4tL \log \frac{1}{s}} \vee \left(\frac{2^\beta \log \frac{1}{s}}{m} \right)^{\frac{1}{\beta}}, \quad s \in (0, 1].$$

Using (38) and (39) for $|x| > c_4\sqrt{t}$, $t > 1$ we obtain

$$\begin{aligned} p_t(x) = \tilde{p}_t * \bar{P}_t(x) &= \int \tilde{p}_t(x-y) \bar{P}_t(dy) \\ &\leq \int c_5[h(t)]^{-d} g(|y-x|) \bar{P}_t(dy) \\ &= c_5[h(t)]^{-d} \int \int_0^{g(|y-x|)} ds \bar{P}_t(dy) \\ &= c_5[h(t)]^{-d} \int_0^1 \int \mathbb{1}_{\{y \in \mathbb{R}^d: g(|y-x|) > s\}} \bar{P}_t(dy) ds \\ &= c_5[h(t)]^{-d} \int_0^1 \bar{P}_t(B(x, g^{-1}(s))) ds \\ &\leq c_5 c_6 [h(t)]^{-d} \left(\int_{g(|x|/2)}^1 t f(|x|/4) g^{-1}(s) ds + \int_0^{g(|x|/2)} ds \right) \\ &\leq c_7 t^{-d/2} \left(t^{3/2} f(|x|/4) + e^{-\left(\frac{|x|^2}{16tL} \wedge \frac{m|x|^\beta}{4^\beta}\right)} \right) \\ &\leq c_8 t^{-d/2} \left(e^{\frac{-|x|^2}{16tL}} + (1 + t^{3/2}|x|^{-1-\alpha}(1+|x|)^\kappa) e^{\frac{-m|x|^\beta}{4^\beta}} \right) \\ &= c_8 t^{-d/2} \left(e^{\frac{-|x|^2}{16tL}} + \left(1 + \left(\frac{\sqrt{t}}{|x|} \right)^3 |x|^{2-\alpha}(1+|x|) \right)^\kappa e^{\frac{-m|x|^\beta}{4^\beta}} \right) \\ &\leq c_9 t^{-d/2} \left(e^{\frac{-|x|^2}{16tL}} + e^{\frac{-m|x|^\beta}{2.4^\beta}} \right), \end{aligned}$$

which yields (3). If $|x| \leq c_4\sqrt{t}$ then (3) follows directly from (7).

Taking $F(s) = c_{10}(s \wedge s^{\alpha-1})$ for $\alpha > 1$ and $F(s) = c_{11}s$ for $\alpha \leq 1$ we obtain $F(s) \leq \Psi(s)/s$ for $\alpha > 1$ and $s > 0$, and for $\alpha \leq 1$ and $s \in (0, 1)$. From Lemma 3.2 we get

$$(40) \quad p_t(x) \geq c_{12} t^{-d/2} e^{-c_{13}|x|^2/t}, \quad |x| < c_{14}t, \quad t > t_0.$$

From Proposition 2.1 it follows that

$$(41) \quad p_t(x) \geq c_{14} t^{1-d/2} \nu(B(x, c_{15}\sqrt{t})), \quad |x| > \eta\sqrt{t}, \quad t > t_0,$$

and (4) follows from (40) and (41).

If (5) holds and $|x| \geq \eta\sqrt{t}$ then $\nu(B(x, c_{15}\sqrt{t})) \geq c_{16}t^{d/2}e^{-c_{17}|x|^\beta}$ for some constants c_{16}, c_{17} . Furthermore, we have $c_{17}|x|^\beta < c_{18}|x|^2/t$ for $|x| \geq c_{14}t$, and (6) follows from (40) and (41) for $\eta\sqrt{t} < |x| < c_{14}t$, from (41) for $|x| \geq c_{14}t$ and from (7) for $|x| \leq \eta\sqrt{t}$. \square

6 High intensity of small jumps

We consider now an interesting example, which has been studied in [23] and [17]. The exact estimates of transition densities for small x and small t are still unreachable in this case, but using the above results we can improve them significantly. Let ν be a Lévy measure such that

$$(42) \quad \nu(dx) \asymp |x|^{-d-2} \left[\log \left(\frac{2}{|x|} \right) \right]^{-\beta} dx, \quad |x| < 1,$$

where $\beta > 1$.

This assumption gives following properties of the corresponding semigroup.

LEMMA 6.1. *If the Lévy measure ν satisfies (1) and (42) then*

$$(43) \quad \Phi(\xi) \asymp \Psi(|\xi|) \asymp |\xi|^2 [\log(2|\xi|)]^{1-\beta}, \quad |\xi| \geq 1.$$

Furthermore,

$$h(t) \asymp t^{1/2} \left[\log \left(\frac{2}{t} \right) \right]^{(1-\beta)/2}, \quad t < 1,$$

and **(A1)** holds with $t_p = 1$.

Proof. For $|\xi| > 1$ by (2) we have

$$\begin{aligned} \Phi(\xi) &\leq 2|\xi|^2 \int_{|y| \leq 1/|\xi|} |y|^2 \nu(dy) + 2 \int_{|y| > 1/|\xi|} \nu(dy) \\ &\leq c_1 |\xi|^2 \int_0^{\frac{1}{|\xi|}} r^{-1} \left[\log \frac{2}{r} \right]^{-\beta} dr + c_1 \int_{\frac{1}{|\xi|}}^1 r^{-3} \left[\log \frac{2}{r} \right]^{-\beta} dr \\ &\quad + 2\nu(B(0, 1)^c) \\ &= c_1 |\xi|^2 \int_{\log(2|\xi|)}^\infty s^{-\beta} ds + \frac{c_1}{4} \int_{\log 2}^{\log(2|\xi|)} e^{2s} s^{-\beta} ds + c_2 \\ &\leq c_3 |\xi|^2 [\log(2|\xi|)]^{1-\beta}, \end{aligned}$$

since $\int_{\log 2}^x e^{2s} s^{-\beta} ds \leq c_4 e^{2x} x^{-\beta+1}$, for $x > \log 2$. Similarly, we obtain

$$\begin{aligned}
\Phi(\xi) &= \int (1 - \cos(\xi \cdot y)) \nu(dy) \\
&\geq c_5 \int_{|y| \leq 1/|\xi|} |\xi \cdot y|^2 \nu(dy) \\
&\geq c_6 |\xi|^2 \int_0^{\frac{1}{|\xi|}} r^{-1} \left[\log \frac{2}{r} \right]^{-\beta} dr \\
&= c_7 |\xi|^2 [\log(2|\xi|)]^{1-\beta}, \quad |\xi| > 1,
\end{aligned}$$

and (43) is proved.

For $s > 1$, let $g(s) = s^2 [\log(2s)]^{1-\beta}$. The function g is increasing on $[s_\beta, \infty)$ for some constant $s_\beta \geq 1$, depending on β , so there exists an increasing inverse function $g^{-1} : [g(s_\beta), \infty) \rightarrow [s_\beta, \infty)$. We let $\eta(r) = \left(r [\log(2r)]^{\beta-1} \right)^{1/2}$ for $r > 1$. Then there exists r_β such that for $r > r_\beta$ we have

$$\begin{aligned}
g(\eta(r)) &= r (\log(2r))^{\beta-1} \left[\log \left(2r^{1/2} (\log(2r))^{(\beta-1)/2} \right) \right]^{1-\beta} \\
&= r (\log(2r))^{\beta-1} \left[\frac{1}{2} \log(4r) + \frac{1}{2} (\beta-1) \log \log(2r) \right]^{1-\beta} \\
&= r \left[\frac{\log(4r) + (\beta-1) \log \log(2r)}{2 \log(2r)} \right]^{1-\beta} \\
&\asymp r.
\end{aligned}$$

This shows that $g^{-1}(r) \asymp \eta(r)$ for $r > r_\beta$, and since $h(t) = 1/\Psi^{-1}(1/t) \asymp 1/g^{-1}(1/t)$, it follows that

$$h(t) \asymp t^{1/2} \left[\log \left(\frac{2}{t} \right) \right]^{\frac{1-\beta}{2}}, \quad t \in (0, 1).$$

Furthermore, it follows also that there exists constant $c_* > 1$ such that

$$g^{-1}(2r) \leq c_* g^{-1}(r), \quad r > g(s_\beta),$$

and this, for $t < 1/g(s_\beta)$, yields

$$\begin{aligned}
\int e^{-t\Phi(\xi)} |\xi| d\xi &\leq \int_{|\xi| < g^{-1}(1/t)} |\xi| d\xi + \int_{|\xi| \geq g^{-1}(1/t)} e^{-tc_8 g(|\xi|)} |\xi| d\xi \\
&\leq c_9 \left((g^{-1}(1/t))^{d+1} + \int_{g^{-1}(1/t)}^{\infty} e^{-c_8 t g(s)} s^d ds \right) \\
&\leq c_{10} \left(h(t)^{-d-1} + \sum_{k=0}^{\infty} \int_{g^{-1}(2^k/t)}^{g^{-1}(2^{k+1}/t)} e^{-c_8 t g(s)} s^d ds \right) \\
&\leq c_{10} \left(h(t)^{-d-1} + \sum_{k=0}^{\infty} e^{-c_8 2^k} \frac{1}{d+1} \left(g^{-1} \left(\frac{2^{k+1}}{t} \right) \right)^{d+1} \right) \\
&\leq c_{10} \left(h(t)^{-d-1} + \frac{1}{d+1} \sum_{k=0}^{\infty} e^{-c_8 2^k} c_*^{(k+1)(d+1)} \left(g^{-1} \left(\frac{1}{t} \right) \right)^{d+1} \right) \\
&\leq c_{11} h(t)^{-d-1}.
\end{aligned}$$

□

Except of (1) we do not assume here anything on the behavior of ν outside of the ball $B(0, 1)$. However it follows easily from (2) that for every ν satisfying (42) there exists c_1 such that $\Psi(s) \geq c_1 s^2$, for $s < 1$, and so the condition (11) holds with $F(s) = c_2 s [\log(e^{\beta-1} + s)]^{1-\beta}$. Furthermore, $F^{-1}(s) \asymp s [\log(e^{\beta-1} + s)]^{\beta-1}$ for $s > 0$ and from Lemma 3.1 we obtain

$$p_t(x) \geq c_3 h(t)^{-d} e^{-c_4(|x|^2/t) \log(e^{\beta-1} + c_5|x|/t)^{\beta-1}},$$

for $t > 0$, $x \in \mathbb{R}^d$. If we consider $t < 1$ and $|x| < 1$, then we get

$$p_t(x) \geq c_3 h(t)^{-d} e^{-c_4(|x|^2/t) \log(e^{\beta-1} + c_5/t)^{\beta-1}} \geq c_6 h(t)^{-d} e^{-c_7(|x|/h(t))^2}.$$

Combining the estimate with Proposition 2.1 and (7) we obtain

$$p_t(x) \geq c_8 \min \left\{ t^{-d/2} \left(\log \frac{2}{t} \right)^{d(\beta-1)/2}, \frac{t}{|x|^{d+2} \left(\log \frac{2}{|x|} \right)^{\beta}} + h(t)^{-d} e^{-c_7(|x|/h(t))^2} \right\},$$

for $|x| < 1$ and $t < 1$.

If the Lévy measure ν has a density which is bounded on $B(0, 1)^c$ then from Theorem 1 in [17] we obtain the following upper estimate.

$$p_t(x) \leq c_8 \min \left\{ t^{-d/2} \left(\log \frac{2}{t} \right)^{d(\beta-1)/2}, \frac{t}{|x|^{d+2} \left(\log \frac{2}{|x|} \right)^{\beta}} + h(t)^{-d} e^{\frac{-c_9|x|}{h(t)} \log \left(1 + \frac{c_{10}|x|}{h(t)} \right)} \right\},$$

for $|x| < 1$ and $t < 1$.

We see that we do not have sharp both sides estimates in this case, however the new results of [24] obtained for subordinated Brownian motion (contained in the case of $\beta = 2$ here) show that the lower estimate is optimal (note that $h(t) \leq |x|$ under the assumption $t\Psi(|x|^{-1}) \leq 1$ given in [24]).

7 Appendix

Proof of Proposition 2.1. Similarly as in the proof of Theorem 1.4 we consider \tilde{p}_t^r and \bar{P}_t^r , noting that (34) also holds and hence the densities \tilde{p}_t^r exist for every $t \in (0, t_p)$ and $r > 0$. First we will prove that there exist constants $c_1 = c_1(d)$, $c_2 = c_2(d, M_0)$, $c_3 = c_3(d)$ such that for every $a \in (0, 1]$ we have

$$(44) \quad \tilde{p}_t^{h(at)}(y) \geq c_1 (h(t))^{-d},$$

provided $|y| \leq c_2 e^{-c_3/a} h(t)$, $t \in (0, t_p)$.

By symmetry of ν we have

$$\mathcal{F}(\tilde{p}_t^{h(at)})(\xi) \geq |\mathcal{F}(p_t)(\xi)|, \quad \xi \in \mathbb{R}^d, t \in (0, t_p),$$

and this and Lemma 4 in [17] yield

$$\begin{aligned} \tilde{p}_t^{h(at)}(0) &\geq (2\pi)^{-d} \int e^{-t \operatorname{Re}(\Phi(\xi))} d\xi \\ &\geq c_4 (h(t))^{-d}, \quad t \in (0, t_p), \end{aligned}$$

where the constant c_4 depends only on d . It follows from (34) that

$$|\mathcal{F}(\tilde{p}_t^{h(at)})(\xi)| \leq |\mathcal{F}(p_t)(\xi)| e^{2t\nu(B(0, h(at)))^c}$$

and since by (2) we have $\nu(B(0, r)^c) \leq (1/L_0)\Psi(1/r)$, we obtain

$$|\mathcal{F}(\tilde{p}_t^{h(at)})(\xi)| \leq e^{-t \operatorname{Re}(\Phi(\xi))} e^{2t\Psi(1/h(at))/L_0} = e^{-t \operatorname{Re}(\Phi(\xi))} e^{2/(aL_0)},$$

and for every $j \in \{1, \dots, d\}$ by (A1) we get

$$\begin{aligned} \left| \frac{\partial \tilde{p}_t^{h(at)}}{\partial y_j}(y) \right| &= \left| (2\pi)^{-d} \int (-i)\xi_j e^{-iy \cdot \xi} \mathcal{F}(\tilde{p}_t^{h(at)})(\xi) d\xi \right| \\ &\leq (2\pi)^{-d} e^{2/(aL_0)} \int_{\mathbb{R}^d} e^{-t \operatorname{Re}(\Phi(\xi))} |\xi| d\xi \\ &\leq c_6 e^{2/(aL_0)} (h(t))^{-d-1}, \end{aligned}$$

with $c_6 = c_6(d, M_0)$. It follows that

$$\tilde{p}_t^{h(at)}(y) \geq c_4 (h(t))^{-d} - dc_6 e^{2/(aL_0)} (h(t))^{-d-1} |y| \geq \frac{1}{2} c_4 (h(t))^{-d},$$

provided $|y| \leq \frac{c_4}{2dc_6} e^{-2/aL_0} h(t)$, which clearly yields (44).

Let $a \in (0, 1)$ and $t \in (0, t_p)$. For $r > 0$, $|x| > r + h(at)$ by (35) and (2) we get

$$\bar{P}_t^{h(at)}(B(x, r)) \geq e^{-1/aL_0} t \bar{\nu}_{h(at)}(B(x, r)) = e^{-1/aL_0} t \nu(B(x, r)).$$

This and (44) for $x \in (0, t_p)$ yield

$$\begin{aligned} p_t(x) &= \tilde{p}_t^{h(at)} * \bar{P}_t^{h(at)}(x) \\ &= \int \tilde{p}_t^{h(at)}(x - z) \bar{P}_t^{h(at)}(dz) \\ &\geq c_1 \int_{|z-x| < c_2 e^{-c_3/a} h(t)} (h(t))^{-d} \bar{P}_t^{h(at)}(dz) \\ &= c_1 (h(t))^{-d} \bar{P}_t^{h(at)}(B(x, c_2 e^{-c_3/a} h(t))) \\ &\geq c_1 e^{-1/aL_0} t (h(t))^{-d} \nu(B(x, c_2 e^{-c_3/a} h(t))), \end{aligned}$$

provided $|x| > h(at) + c_2 e^{-c_3/a} h(t)$. Using the fact that $h(at)/h(t) \leq \sqrt{2a/L_0}$ for $a < L_0/2$ and $t > 0$ (see the proof of Lemma 11 in [17]) we choose $a \in (0, 1)$ such that $h(at)/h(t) + c_2 e^{-c_3/a} \leq \eta$ and we obtain (8). \square

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